Int. J. Heat Mass Transfer. Vol. 31, No. 9, pp. 1861–1872, 1988
Printed in Great Britain

0017-9310/88 $3.00+0.00

Thermal constriction resistance with convective boundary conditions—1. Half-space contacts

T. F. LEMCZYK and M. M. YOVANOVICH
Microelectronics Heat Transfer Laboratory, Department of Mechanical Engineering, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

(*Received* 27 *April* 1987 *and in final form* 12 *February* 1988)

Abstract—Analytical thermal contact analysis in the past has largely been restricted to idealized boundary conditions on the contact surface. Recently, Gladwell *et al.* (*Q. J. Mech. Appl. Math.* **36**(3), 387–401 (1983)) have outlined the efficient evaluation of the resulting integro-differential equation for four basic axisymmetric problems with uniform convective boundary conditions. This paper outlines in non-dimensional form, the variation of the thermal constriction resistance with Biot number for four mixed problem types on a homogeneous half-space. In addition, the thermal analysis is extended to include non-uniform flux and non-uniform convective conditions. In each case, the constriction resistance is given as a compact expression, and for several cases accurate but simpler correlations are presented.

1. INTRODUCTION

AN AXISYMMETRIC boundary-value problem occurs in the classical theory of thermoelasticity when an axisymmetric heated punch, which may be circular or annular, flat or curved, contacts an elastic half-space. The uncoupled thermal problem is largely concerned with establishing the temperature field in the contact zone, satisfying the surface boundary conditions. Having determined the temperature field, one may proceed as required to evaluate stresses and displacements for the dependent elastic problem, as discussed in ref. [1].

The mathematical problem is not new, and has been studied exhaustively in the older literature [2]. For definitive treatments, see Sneddon [3], Lowengrub and Sneddon [4], Collins [5] and Sneddon [6]. However, when the surface boundary conditions are of a mixed Robin (convective) type, the resulting sets of integral equations have largely been solved numerically, as discussed by Poddubny [7], Kuz'min [8], Pavlovskii [9] and Linz [10]. These numerical techniques include the finite element [11], finite difference, and boundary element methods, and more recently, the method of moments in ref. [12]. Negus and Yovanovich [13] used the method of optimized images to examine three-dimensional conduction problems for arbitrary contacts on arbitrary flux tubes, however, with Dirichlet and Neumann boundary conditions only. Huang [14] looked at several two-dimensional, transient problems using the Weiner–Hopf technique.

Several recent investigations [15–17], have shown that application of orthogonal polynomials to mixed boundary-value problems in elasticity, yields useful and simple solutions. Gladwell *et al.* [18] showed hence that axisymmetric thermal problems with convective boundary conditions could be reduced to a linear, infinite set of algebraic equations. Extending this analytical integral transform procedure to include non-uniform convection coefficients and non-uniform flux distributions, this work is an extensive examination on the behaviour of the thermal constriction resistance of circular contacts on a homogeneous half-space. This thereby extends the study of thermal constriction resistance, as studied in ref. [19], to include convective boundary conditions. In the sequel Part 2, similar extensive examination is conducted for contacts on a layered half-space.

2. PROBLEM STATEMENT

Steady-state heat conduction from a circular contact to an isotropic half-space is governed by Laplace's harmonic equation for the medium

$$\nabla^2\Theta = 0 \tag{1}$$

where Θ is the temperature excess field $(T-T_\infty)$. This harmonic field may be denoted, in non-dimensional circular cylinder coordinates ($\rho = r/a$, $\zeta = z/a$, $a =$ contact radius), by using the Hankel integral transform

$$\Theta(\rho,\zeta) = \mathscr{H}_0[\xi^{-1}A(\xi)\exp(-\xi\zeta);\rho] \tag{2}$$

suitable only for the half-space formulation. The four types of mixed convective boundary-value problems to be studied as in ref. [18], are outlined as follows, in non-dimensional form:

NOMENCLATURE

a	contact radius dimension
a_n	Fourier series expansion coefficient and solution vector
A	standard system matrix
$A(\xi)$	Hankel transformed temperature function
b_n, b'_n	Fourier series expansion coefficients
B	coefficient matrix in external convection problems
c	non-uniform flux and convection distribution parameter
$c_1, \ldots, c_4$	correlation parameters
c_m, d_m	Legendre series expansion coefficients
d	non-uniform flux and convection distribution parameter
D	symmetric coefficient matrix
$f(x)$	Fourier transform function
f_{m+n}	Fourier series coefficient
F	symmetric coefficient matrix
g_n	Legendre series expansion coefficient and vector
h, h_1, h_2	convection coefficients
h_{m+n}	Fourier series coefficient
H_i, H_1, H_2	dimensionless Biot numbers, $h_i a/k$
I	identity matrix
k	thermal conductivity
k_m, l_m	Legendre series coefficients, $c_m - e_m$, $c_m - d_m$
m, n	integer constants
n_T	truncation value of system of equations
N	diagonal coefficient matrix
P_n	Legendre polynomial of degree n
$q(\rho)$, $q_0(\rho)$, q_0	heat flux functions, uniform heat flux
Q, Q^*	total heat flux through contact, total flux, Q/ak
r	radial coordinate
R_c	thermal constriction resistance
T, T_∞	system temperature, reference temperature
U	Heaviside unit step function
v_m	Legendre series expansion coefficient
w	Legendre polynomial argument, $2p^2-1$
x	Fourier transformed coordinate
z	depth coordinate.

Greek symbols

α_n, β_n	Fourier series expansion coefficients
$\delta_{n,0}$	delta function, equals unity only for $n = 0$
ζ	dimensionless depth coordinate, z/a
Θ, $\bar{\Theta}_c$, Θ_0, Θ_b	temperature excess, mean contact temperature excess, specified base and contact temperature excesses
θ, ϕ	angular coordinate transformations
μ, ν	general integers
ξ	transformed radial coordinate
π	constant, 3.14159265 . . .
ρ	dimensionless radial coordinate, r/a
Ψ_c	dimensionless constriction factor, $4akR_c$.

Other symbols

$\mathscr{A}_1$, $\mathscr{A}_2$	Abel integral operator transforms
∂	partial derivative operation
$\mathscr{F}_c$, $\mathscr{F}_s$	Fourier cosine and sine transform operators
$\mathscr{H}_\nu$	Hankel transform of order ν
∇	Laplacian operator in cylindrical polar coordinates.

(i)

$$\frac{\partial\Theta}{\partial\zeta} - H_1(\rho)\Theta = -H_1(\rho)\Theta_b, \quad \rho < 1 \tag{3}$$

$$\frac{\partial\Theta}{\partial\zeta} = 0, \quad \rho > 1 \tag{4}$$

(ii)

$$\frac{\partial\Theta}{\partial\zeta} - H_1(\rho)\Theta = -H_1(\rho)\Theta_b, \quad \rho < 1 \tag{5}$$

$$\Theta = 0, \quad \rho > 1 \tag{6}$$

(iii)

$$\frac{\partial\Theta}{\partial\zeta} = -q_0(\rho), \quad \rho < 1 \tag{7}$$

$$\frac{\partial\Theta}{\partial\zeta} - H_2(\rho)\Theta = 0, \quad \rho > 1 \tag{8}$$

(iv)

$$\Theta = \Theta_0, \quad \rho < 1 \tag{9}$$

$$\frac{\partial\Theta}{\partial\zeta} - H_2(\rho)\Theta = 0, \quad \rho > 1. \tag{10}$$

We note that H_1 and H_2 are the dimensionless Biot numbers, $h_i a/k$, corresponding to axisymmetric distributions in the internal ($\rho < 1$) and external ($\rho > 1$) regions, respectively. The dimensional flux $q_0(\rho)$, is defined as $q(\rho)a/k$. Θ_b and Θ_0 are respectively the specified base and contact temperature excesses. The systems to be studied are illustrated in Fig. 1.

The thermal constriction resistance of the circular contact spot on the half-space [20] is defined as

$$R_c = \frac{\bar{\Theta}_c}{Q} \tag{11}$$

where the mean contact temperature rise $\bar{\Theta}_c$, and total heat flux over the contact, Q, are defined in dimensional and non-dimensional coordinates as

$$\bar{\Theta}_c = \frac{1}{\pi a^2}\int_0^a \Theta(r,0)2\pi r\,\mathrm{d}r \tag{12}$$

$$= 2\int_0^1 \Theta(\rho,0)\rho\,\mathrm{d}\rho \tag{13}$$

$$Q = -\int_0^a \frac{\partial\Theta(r,0)}{\partial z}2\pi r\,\mathrm{d}r \tag{14}$$

$$= -2\pi ka\int_0^1 \frac{\partial\Theta(\rho,0)}{\partial\zeta}\rho\,\mathrm{d}\rho. \tag{15}$$

In Cases (iii) and (iv), one of these quantities is easily evaluated, that is

$$\text{(iii)} \Rightarrow Q = 2\pi ka\int_0^1 q_0(\rho)\rho\,\mathrm{d}\rho \tag{16}$$

$$\text{(iv)} \Rightarrow \bar{\Theta}_c = \Theta_0. \tag{17}$$

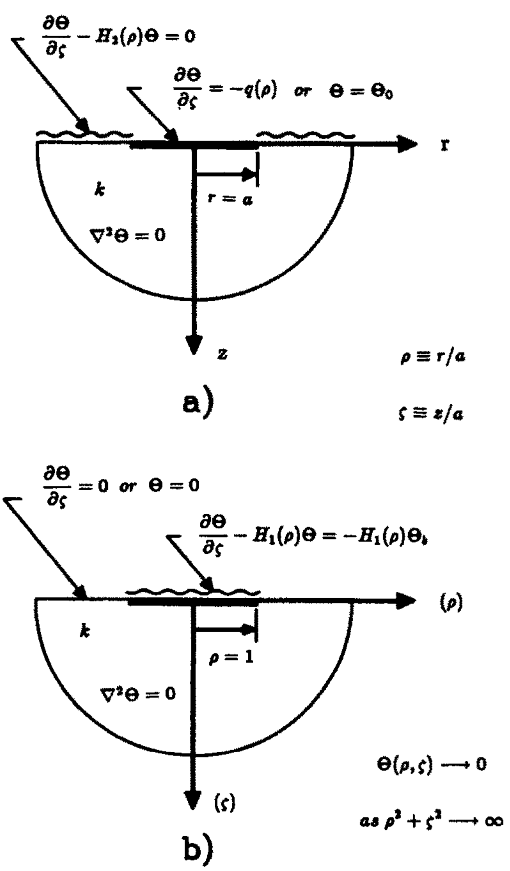

FIG. 1. Half-space contacts with convective boundaries.

Finally, we define the *dimensionless thermal constriction resistance factor*

$$\Psi_c = 4akR_c. \tag{18}$$

Equations (3)–(10) are problem sets of dual integral equations when posed using Hankel transforms. They may be reduced to a single integro-differential equation as discussed in ref. [18], by employing appropriate Abel and Fourier transform relations. The integro-differential equation may then be reduced to an infinite linear set of algebraic equations by employing suitable Fourier expansions.

3. LIMITING SOLUTIONS

In order to validate asymptotically the solutions when H_1 or H_2 approach zero (insulated condition) or approach infinity (isothermal condition), it is important to summarize the limiting cases that are necessary.

Reference [19] has previously determined that when the external boundary is *insulated*, the evaluation of thermal constriction resistance for isothermal and constant flux (*isoflux*) contacts, is straightforward, using the Hankel representation (2). In addition, three other cases are shown in Table 1, including the two limiting problems for a symmetric contact flux distribution of the parabolic form

$$q_0(\rho) = q_0(c\rho^2 + d). \tag{19}$$

We note that Cases (iv) and (v) from Table 1, represent bounds on the thermal constriction resistance of a symmetric flux contact (19) with a *uniform* external Biot number H_2. They respectively represent the *upper* and *lower* bounds on the constriction resistance as $H_2 \to 0$ and $H_2 \to \infty$.

4. PROBLEMS WITH A UNIFORM HEAT TRANSFER COEFFICIENT

By assuming a uniform convection coefficient on the contact zone or external region, the four basic problems in equations (3)–(10) may be reduced to the *same* infinite linear set of equations. Thus for clarity, we outline here the problem with a uniform contact conductance with external insulation, as defined in equations (3) and (4). In Hankel operator form, the dual integral equations become

$$\mathscr{H}_0[A(\xi);\rho] + H_1\mathscr{H}_0[\xi^{-1}A(\xi);\rho] = H_1\Theta_b, \quad \rho < 1 \tag{20}$$

$$\mathscr{H}_0[A(\xi);\rho] = 0, \quad \rho > 1 \tag{21}$$

where H_1 and Θ_b are uniform values. Condition (21) necessitates that we may represent

$$\mathscr{F}_c[A(\xi);x] = (\tfrac{1}{2}\pi)^{1/2}f(x)U(1-x) \tag{22}$$

using the Abel operator $\mathscr{A}_2$ [18]. U is the Heaviside unit function, defined here as unity for $x < 1$ and zero for $x > 1$. With this representation, we may reduce

Table 1. Summary of limiting cases

	Boundary conditions on surface $\zeta = 0$, $\rho < 1$	$\rho > 1$	Mean contact temperature, $\bar{\Theta}_c$	Total heat flux, $Q^* = Q/ak$	Constriction factor, $4\bar{\Theta}_c/Q^*$
1	$\Theta = \Theta_0$	$\frac{\partial\Theta}{\partial\zeta} = 0$	Θ_0	$\frac{\Theta_0}{4}$	1
2	$\frac{\partial\Theta}{\partial\zeta} = -q_0$	$\frac{\partial\Theta}{\partial\zeta} = 0$	$\frac{8q_0}{3\pi}$	πq_0	$\frac{32}{3\pi^2}$
3	$\frac{\partial\Theta}{\partial\zeta} = -q_0$	$\Theta = 0$	$\frac{4q_0}{3\pi}$	πq_0	$\frac{16}{3\pi^2}$
4	$\frac{\partial\Theta}{\partial\zeta} = -q_0(\rho)$	$\frac{\partial\Theta}{\partial\zeta} = 0$	$\frac{8q_0}{45\pi}(7c+15d)$	$\frac{\pi q_0}{2}(c+2d)$	$\frac{64}{45\pi^2}\left(\frac{7c+15d}{c+2d}\right)$
5	$\frac{\partial\Theta}{\partial\zeta} = -q_0(\rho)$	$\Theta = 0$	$\frac{4q_0}{15\pi}(2c+5d)$	$\frac{\pi q_0}{2}(c+2d)$	$\frac{32}{15\pi^2}\left(\frac{2c+5d}{c+2d}\right)$

the dual equations (20) and (21) to the single Abel-type integro-differential equation

$$-\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_\rho^1 \frac{xf(x)\,\mathrm{d}x}{(x^2-\rho^2)^{1/2}} + H_1\int_0^\rho \frac{f(x)\,\mathrm{d}x}{(\rho^2-x^2)^{1/2}} = -H_1\Theta_b. \quad (23)$$

Integro-differential equations such as equation (23) have been exhaustively studied through the years by numerous researchers. They basically require some approximation to the function $f(x)$ and suitable selection of collocation points to ensure convergence. An excellent discussion of various procedures is given by Ioakimidis and Theocaris [21], who have summarized an excellent methodology based on Chebyshev series approximations for $f(x)$. It should be noted that equation (23) may also be represented as a Fredholm integral equation of the second kind, for which numerous solution procedures exist, as noted in earlier work [22].

We note however, from refs. [18, 23], by using two simultaneous Fourier expansions for $f(x)$, that is

$$f(x) = F(\theta) = \sum_{n=0}^{\infty} a_n \sin(2n+1)\theta, \quad 0 < \theta < \pi/2 \quad (24)$$

$$\sin\theta\, F(\theta) = \sum_{n=0}^{\infty} b_n \sin(2n+1)\theta, \quad 0 < \theta < \pi/2 \quad (25)$$

and with the transformations $x = \cos\theta$, $\rho = \cos\phi$, then we may obtain (after employing Legendre polynomial expansions) the system

$$a_n + H_1(2n+1)^{-1}b_n = -g_n; \quad n = 0, 1, 2, \ldots \quad (26)$$

The b_n are related to the a_n through the theory of Fourier series as

$$b_n = \sum_{m=0}^{\infty} d_{m,n}a_m \quad (27)$$

where the $d_{m,n}$ are integral functions of m and n, and as noted in ref. [18], they may be reduced in this case to closed form expressions. Furthermore, the remaining problems defined by equations (5)–(10), may also be reduced to the same infinite set (26), which satisfies regularity conditions [24], and therefore may be approximated by truncation. Further details on the derivation of the half-space integro-differential equations, may be found in refs. [18, 25]. In summary, the four problems with *uniform* Biot numbers H_i, may all be reduced to the matrix system

$$\{[I]+H_1[N][D]\}a_n = g_n \quad (28)$$

which may be solved quite easily for the a_n, and the accuracy of the solution will depend on the truncation value n_T used. The matrix equations for each system are summarized in the Appendix.

The necessary quantities of mean contact temperature and total heat flux through the contact (equations (12) and (14)) may be easily expressed in terms of the solution vector a_n in equation (28), as found in ref. [25]. For the imperfect contact problems with uniform contact conductance, we may also express $\bar{\Theta}_c$ in terms of Q, by multiplying the contact boundary equation by ρ and integrating from $\rho = 0$ to 1. Thus, we may find

$$\bar{\Theta}_c = \Theta_b - \frac{Q^*}{H_1\pi} \quad (29)$$

where

$$Q^* \equiv Q/ak.$$

We now summarize the solution quantities for the four basic problems. For the imperfect contacts with uniform Biot number H_1, the mean contact temperature is given by equation (29), and the total heat flux Q^* is denoted by

$$Q^* = \frac{\pi^2}{2}a_0 \quad (30)$$

where a_0 is the first solution entry of equation (28). Thus for both problems defined by equations (3)–(6), we obtain

$$\Psi_c = 4\left(\frac{\Theta_b}{Q^*} - \frac{1}{H_1\pi}\right). \quad (31)$$

For the third basic problem, with a uniform flux q_0 over the contact, and uniform external Biot number H_2, we have

$$\bar{\Theta}_c = \frac{4q_0}{3\pi} + \sum_{n=1}^{\infty} \alpha_n \left[(-1)^{n+1} \times \left\{\pi - 4\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n-1}}{2n-1}\right)\right\} + \frac{4n}{4n^2-1}\right] \quad (32)$$

$$Q^* = \pi q_0, \quad \Psi_c = 4\bar{\Theta}_c/Q^*. \quad (33)$$

Finally, for an isothermal contact with uniform external Biot number H_2, we get

$$\bar{\Theta}_c = \Theta_0, \quad Q^* = 2\pi\left[\Theta_0 + \sum_{n=1}^{\infty} \alpha_n \left(\frac{1}{1-4n^2}\right)\right]. \quad (34)$$

In addition, we must consider the problems defined by equations (7) and (8) when $q_0(\rho)$ takes on the form of equation (19). In this case, the entries of the g_n in equation (26) are slightly modified from the form they have in ref. [18]. These are summarized in the Appendix, and upon solving the matrix equation, we may obtain

$$Q^* = \pi q_0\left(\frac{c}{2} + 2\right), \quad \bar{\Theta}_c = \frac{4q_0}{3\pi}\left(\frac{2}{5}c + d\right) + S_n \quad (35)$$

where S_n is the series term defined previously in equation (32).

5. NON-UNIFORM CONDUCTANCE/CONVECTION COEFFICIENTS

By introducing a *non-uniform* convection or contact conductance coefficient into the basic problems outlined in the previous section, improved insights can be gained in the study of thermal constriction resistance. Since thermal contact conductance has been found in experimental work [26] to depend on the contact pressure, a variable coefficient would more accurately represent the variable pressure distribution which occurs depending upon punch geometry. Additionally, it could account for variable surface parameters such as asperity roughness, over the contact radius. Similarly, a variable *external* convection coefficient more accurately represents the fluid flow model outside the contact, since it is disrupted by the presence of the punch, and therefore can only be non-uniform.

First, for a non-uniform contact conductance, we choose the symmetric form

$$h(\rho) = h_1(c\rho^2 + d) \quad (36)$$

with c and d defined analogously as for the variable flux contact discussed earlier. In the following, we will therefore work with a *redefined* internal Biot number, defined as

$$H_1 \equiv \frac{h_1 a}{k}. \quad (37)$$

Now, the integro-differential equation (23), expressed in terms of a series of Legendre polynomials becomes

$$\sum_{n=0}^{\infty} (2n+1)a_n P_n(2\rho^2 - 1) + H_1(c\rho^2 + d)\sum_{n=0}^{\infty} b_n P_n(2\rho^2 - 1) = \sum_{n=0}^{\infty} (2n+1)g_n P_n(2\rho^2 - 1) \quad (38)$$

where the argument of P_n is $w \equiv 2p^2 - 1$.

The Legendre polynomials satisfy the well-known recurrence relation (8.914.1 in ref. [27])

$$(n+1)P_{n+1}(w) = (2n+1)wP_n(w) - nP_{n-1}(w), \quad n = 1, 2, 3, \ldots \quad (39)$$

Using this, we may suitably approximate the non-linear component in equation (38) by

$$w\sum_{n=0}^{\infty} b_n P_n(w) = \sum_{n=0}^{\infty} b'_n P_n(w) \quad (40)$$

where

$$b'_0 = \tfrac{1}{3}b_1 \quad (41)$$

$$b'_n = \frac{n}{2n-1}b_{n-1} + \frac{n+1}{2n+3}b_{n+1}; \quad n = 1, 2, 3, \ldots \quad (42)$$

Substituting into equation (38), we thus obtain an *approximate* linear set

$$a_n + H_1\left(\frac{c}{2} + d\right)(2n+1)^{-1}b_n + H_1\frac{c}{2}(2n+1)^{-1}b'_n = g_n, \quad n = 0, 1, 2, \ldots \quad (43)$$

The b'_n are related exactly to the b_n by

$$b'_n = [G]b_n \quad (44)$$

where $[G]$ is an $n_T \times (n_T + 1)$ *rectangular* matrix, b' is $(n_T \times 1)$ and b is of size $(n_T + 1) \times 1$. However, for computational purposes, we ultimately wish to represent b_n in terms of the solution vector a_n, through a matrix $[D]$ defined in equation (27). If we approximate $[D]$ to be of rectangular size $(n_T + 1) \times n_T$ (denoted by matrix $[D']$), then the product will be square, i.e.

$$b'_n = [G][D']a_n. \quad (45)$$

For increasing n_T, it was found that this rectangular approximation was stable and gave converging results, since the b_n and a_n both decrease suitably with increasing n. Details on the resulting matrix equation with modified entries can be found in the Appendix. Owing to the non-uniform internal Biot number H_1, we may not use the form (29) to relate the mean contact temperature to total heat flux. Instead, we may directly express the mean contact temperature in terms of the solution vector a_n. Thus from ref. [25], it may be shown that

$$\bar{\Theta}_c = \frac{(-1)^\mu}{2} \sum_{n=0}^{\infty} a_n \left[\frac{1}{1-2n} + \frac{1}{3+2n} \right] \quad (46)$$

where $\mu = 0$ for $\Theta = 0$ on the external boundary, and $\mu = 1$ for $\partial\Theta/\partial\zeta = 0$ externally. For both problems, Q^* is given by equation (30).

Similarly, we can model a variable convection coefficient on the *external* boundary. For this we choose the form

$$h(\rho) = h_2(c\rho^{-2}+d), \quad \rho > 1. \quad (47)$$

The *external* Biot number is now redefined by

$$H_2 \equiv \frac{h_2 a}{k}. \quad (48)$$

The necessary matrix system changes are analogous to those outlined previously for a variable contact conductance, except physically we are now studying non-uniformity on the *external* boundary. It is important to note that they are virtually identical for either the isoflux, isothermal and non-uniform flux-specified contact conditions.

6. PRESENTATION OF RESULTS

The standard system of equations (26) (see Appendix) was solved for all cases using a simple Gaussian elimination routine with scaled partial pivoting [28]. Convergence was governed by the truncation value n_T used for the matrix size, and this was established by specifying an accuracy on the constriction resistance of six significant figures. Generally, a larger truncation value was necessary as the reference Biot number increased, and it was found that a value of $n_T = 50$ gave the accuracy required for Cases (i) and (iii). A slightly larger truncation value was needed for the results in Cases (ii) and (iv), generally requiring $n_T > 50$ but $n_T < 75$. This also applied to the results for non-uniform flux and non-uniform convection conditions. The slower convergence in Cases (ii) and (iv) was expected because as the Biot number increases, the limiting solution in both cases becomes an isothermal contact with $T = 0$ external boundary. This is a physically unrealistic surface condition, causing a potential discontinuity at the radius $r = a$. However, over the range of Biot numbers studied, the constriction resistance should approach smaller and smaller values, but the contact will still remain *non-isothermal.*

Figure 2 illustrates the variation of constriction resistance, via the dimensionless constriction factor, with the Biot number for a uniform contact conductance and insulated external boundary. The upper and lower bounds are respectively the constriction resistance for an isoflux and an isothermal contact. This result was also obtained in ref. [12]. It is interesting to note, that over the entire range of Biot numbers considered, the difference in constriction resistance amounts to only 7.5% between the upper and lower bounds. On the other hand, if the external boundary has the condition $T = 0$ (Fig. 2) the difference between the upper and lower bounds is 68%. It is important to point out that the results for these contacts with contact conductance provide the upper and lower bounds for when the *external* boundary has a uniform convective coefficient. However, this will also hold when we apply a varying external convection coefficient of the form (47). Figure 2 also shows results for when the contact is isoflux or isothermal, and an external uniform convection coefficient is applied. The upper and lower bounds for the isoflux case are respectively the *upper* bounds for the two previous convective contact problems. Here, the percentage difference between the upper and lower limits over the complete range of Biot numbers, is 50%. In other words, the lower bound is precisely half the upper bound. For the isothermal contact, the behaviour is slightly different, and we find that the difference is about 60% over the range of Biot numbers shown. In particular, the upper bound is specified by 1 in Table 1, which is the same as the *lower* bound from Fig. 2 (uniform contact conductance with external insulation).

When we apply a varying flux contact (19) and a uniform external convection coefficient, the upper and lower bounds for all cases are given in Table 1. We note that the constriction resistance is *independent* of the values c and d when $c = -d$ and when $d = 0$. This is also the case when we had an isoflux contact, as in Fig. 2. The resulting distributions for these are similar in form to Fig. 3 except with their appropriate asymptotic limits.

However, when we superimpose flux distributions on top of a uniform flux, as in Fig. 3, we note that the constriction resistance varies for all the cases shown. From Table 1, when $c = -d$, the difference between upper and lower bounds is about 44%, which is smaller than the isoflux case. Also, we note that the upper and lower bounds are slightly *higher* than the 50% for the corresponding isoflux problem. When the flux parameter $d = 0$, from Table 1 we see a difference between upper and lower bounds of 57%, and the bounds are *lower* than the isoflux problem. From Fig. 3, for the given flux variation, we see that the constriction resistance increases steadily as the superimposed flux distribution becomes larger. On the other hand, choosing $d = 10$, and varying $c = 1$, 10, 25, 50, we would find that for a larger applied heat flux, the constriction resistance steadily decreases. These are important features which may not be directly discernible from the flux distributions modelled by equation (19).

For a non-uniform contact conductance with insulated external boundary, in all cases, the lower bound

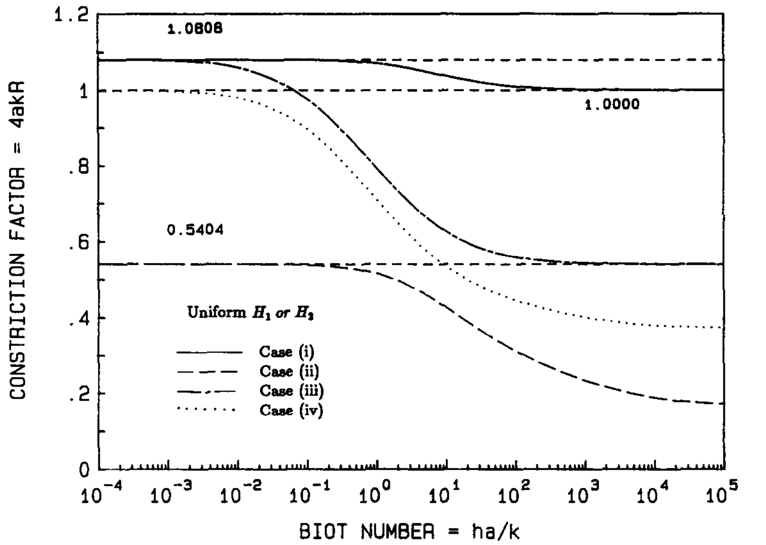

FIG. 2. Constriction factor vs Biot number H_1 or H_2; selected problems.

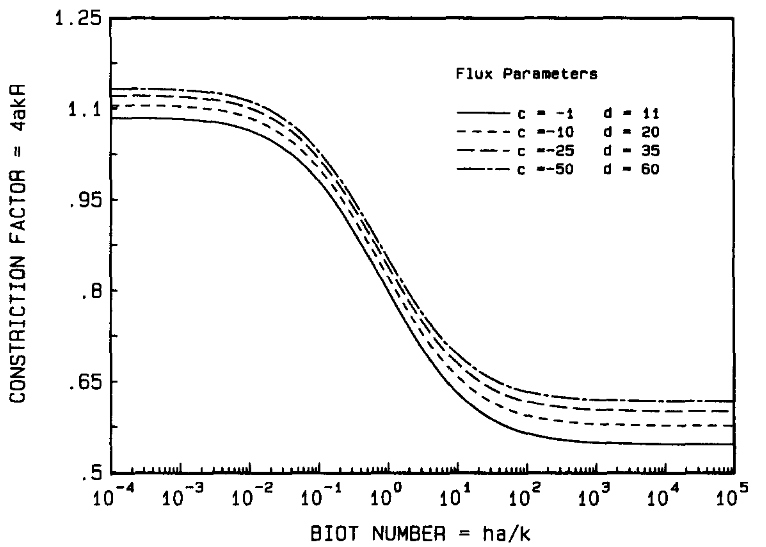

FIG. 3. Constriction factor vs Biot number H_2; varying flux contact with uniform external convection.

is as defined by 1 in Table 1, which is the same as observed for a uniform contact conductance shown in Fig. 2. Similarly, the *upper* bound in all cases corresponds to the constriction resistance of a *non-uniform flux* contact (with the same flux parameters (19) as the conductance parameters (36) given here) with external insulation, as noted by Table 1. Thus we begin to see the direct correspondence and symmetry that is resulting from these various solutions. Furthermore, when we have the conductance parameters defined as $c = -d$ and $d = 0$, particularly useful behaviour can also be noted. If we know the constriction resistance with $c = -1$, $d = 1$, at say, a Biot number of 10, then this will be the *same* constriction resistance obtained for $c = -10$, $d = 10$, at a Biot number of 1. An analogous situation occurs for $d = 0$.

Extensive tabulations are supplied in ref. [25], from which one can predict the constriction resistance accurately for a wide range of non-uniform conductance parameters at various Biot numbers H_1. However, when a non-uniform conductance is superimposed on a given *uniform* conductance, the upper bound is *dependent* on the conductance parameters. Similar results could be shown as for a contact conductance problem with $T = 0$ external boundary.

Next we consider the effect of having a *non-uniform* external convection coefficient of the form (47). For an isoflux contact, the solutions will *always* remain between the upper and lower bounds defined earlier for an isoflux contact with a *uniform* external convection coefficient. However, depending on the convection parameters used, for intermediate Biot numbers H_2 the constriction resistance will vary considerably. These differences are borne out when we compare certain cases as shown in Fig. 4, the largest difference occurring at intermediate Biot numbers H_2. Similar variation of constriction resistance with Biot number occurs for an isothermal contact with non-uniform external convection, as shown in Fig. 5. The upper and lower bounds again remain the same as for the uniform external convection results. If we considered a varying flux contact of the form of equation (19), and a non-uniform external convection coefficient defined by equation (47), then we could anticipate the solutions to lie within the bounds similar to Fig. 3.

Accurate and simple correlations for several cases are listed in Table 2 in a convenient hyperbolic-tangent form adopted from ref. [12]. The hyperbolic tangent is a reasonable function choice since it most closely resembles the form of the distributions generated in the figures. A non-linear least squares curve-fitting routine was used to fit the correlating form C_f over the range $10^{-4} \leqslant H_i \leqslant 10^5$. The largest error associated with each fit occurs generally a decade of Biot numbers to the left or right of the inflection point seen in each distribution (i.e. Biot number of $\sim 10^0$).

As an overview, we note that for all cases studied, the Biot number was varied over the entire range $10^{-4} \leqslant H_i \leqslant 10^5$. However, not all cases approached the upper and lower bounds at the same Biot number. In Fig. 2, the distribution departs from the upper

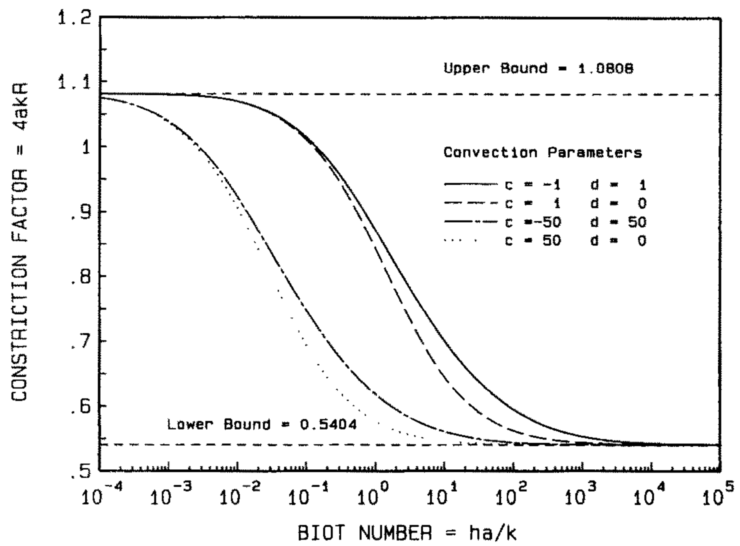

FIG. 4. Constriction factor vs Biot number H_2; isoflux contact with non-uniform external convection.

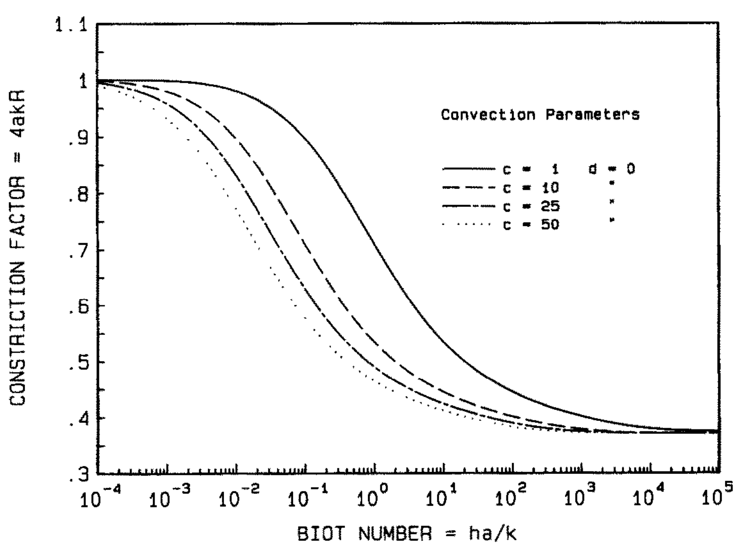

FIG. 5. Constriction factor vs Biot number H_2; isothermal contact with non-uniform external convection.

Table 2. Selected correlations

c_1	c_2	c_3	c_4	$\left\lvert\frac{\Psi_c - C_f}{C_f}\right\rvert \times 100\,(\%)$
	$C_f = c_1 - c_2 \tanh\,(c_3 \ln\,(H_i) - c_4)$			
	Isoflux contact with uniform external convection			
0.81180	0.27274	0.33492	−0.06596	0.26
	Isothermal contact with uniform external convection			
0.71006	0.30890	0.25317	−0.08603	2.5
	Varying flux contacts with uniform external convection			
		$c = -d$		
0.90175	0.25455	0.34061	−0.10743	0.25
		$d = 0$		
0.72177	0.29092	0.33070	−0.029927	0.30
	Isoflux contacts with non-uniform external convection			
		$c = -1,\ d = 1$		
0.81365	0.27407	0.29357	0.22773	0.60
		$c = 1,\ d = 0$		
0.81110	0.27176	0.35965	0.11443	0.20

bound at a Biot number of $\sim 10^{-1}$ (for the contact conductance cases), whereas for external convection cases, it illustrates a departure at a Biot number of $\sim 10^{-3}$. This is particularly important in estimating the error arising from using a limiting (idealized boundary condition) solution which does not account for convection effects. If we also consider non-uniform convection on the contact zone or externally, then these departure limits are altered further. The non-uniform convective models are physically more realistic than a uniform convection coefficient, and therefore, depending on the problem type, this could significantly alter results and better explain discrepancies in experimental investigations. This is particularly important since we do not always know precisely what the boundary conditions are. Experimental experience has shown that the majority of boundary conditions are actually non-linear, and therefore the models developed here for non-uniform convection effects could provide better estimates for bounds, than the idealized conditions previously used.

7. CONCLUSIONS

An analytical examination has been conducted into the variation of thermal constriction resistance with variable Biot number for circular contacts on a half-space. Both uniform and non-uniform flux distributions and convective components have been included. The constriction resistance was found to vary predictably with uniform and non-uniform Biot number, between previously established asymptotic bounds in constriction resistance theory. A common trend observed was that the constriction resistance will *always decrease* as the reference Biot number increases, *regardless* of whether the convection coefficient is uniform or non-uniform, or if the convection is associated externally or on the contact zone (contact conductance). In particular, the graphical results provide a convenient check on what the actual error may be when using idealized limiting boundary conditions in a problem model. Although results were not presented for non-uniform flux contacts with non-uniform external convection conditions, the behaviour of the thermal constriction resistance can be easily predicted based on the results of this work. Additionally, the completely general problem of simultaneous convection conditions on the contact and external boundary, is the subject of an upcoming publication [28]. The bounds for this problem, may finally be established from the results shown in this study.

Acknowledgements—The authors T. F. Lemczyk and M. M. Yovanovich, wish to acknowledge the financial support of the Natural Sciences and Engineering Research Council of Canada under operating grant A7455.

REFERENCES

1. B. A. Boley and J. H. Weiner, *Theory of Thermal Stresses*. Wiley, New York (1960).
2. M. Bocher, *An Introduction to the Study of Integral Equations*. Cambridge University Press, Cambridge (1914).
3. I. N. Sneddon, The elementary solution of dual integral equations, *Proc. Glasg. Math. Ass.* **4**, 108–110 (1959).
4. M. Lowengrub and I. N. Sneddon, An axisymmetric boundary value problem of mixed type for a half-space, *Proc. Edinb. Math. Soc.* **13**, 39–46 (1961).
5. W. D. Collins, On the solution of some axisymmetric boundary value problems by means of integral equations, *Proc. Edinb. Math. Soc.* **13**, 235–246 (1963).
6. I. N. Sneddon, *Mixed Boundary Value Problems in Potential Theory*. North-Holland, Amsterdam (1966).
7. G. V. Poddubny, Heat conduction problem in a homogeneous half-space, *J. Engng Phys.* **4**(5), 27–32 (1961).
8. Yu. N. Kuz'min, Some axially symmetric problems in heat flow with mixed boundary conditions, *Soviet Phys. Tech. Phys.* **11**(2), 169–173 (1966).
9. R. A. Pavlovskii, An approximate solution for a stationary heat-conduction problem in a half-space with mixed

boundary conditions, *J. Engng Phys.* **14**(2), 282–287 (1968).
10. P. Linz, The numerical solution of certain dual integral equations, *J. Engng Math.* **3**(3), 245–249 (1969).
11. G. E. Schneider, Thermal constriction resistance due to arbitrary contacts on a half-space: numerical solution of the Dirichlet problem, AIAA Paper No. 78-270, 2nd AIAA/ASME Thermophysics and Heat Transfer Conference (1978).
12. K. A. Martin, M. M. Yovanovich and Y. L. Chow, Method of moments formulation of thermal constriction resistance of arbitrary contacts, AIAA Paper No. 84-1745, AIAA 19th Thermophysics Conference, Colorado (1984).
13. K. J. Negus and M. M. Yovanovich, Application of the method of optimized images to steady three-dimensional conduction problems, ASME Paper No. 84-WA/HT-110 (1984).
14. S. C. Huang, Unsteady-state heat conduction in semi-infinite regions with mixed-type boundary conditions, *J. Heat Transfer* **107**, 489–491 (1985).
15. G. M. L. Gladwell and A. H. England, Orthogonal polynomial solutions to some mixed boundary-value problems in elasticity theory, *Q. J. Mech. Appl. Math.* **30**(2), 175–185 (1977).
16. A. H. England and R. Shail, Orthogonal polynomial solutions to some mixed boundary-value problems in elasticity—II, *Q. J. Mech. Appl. Math.* **30**(4), 397–414 (1977).
17. A. H. England, Love's integral and other relations between solutions to mixed boundary-value problems in potential theory, *J. Aust. Math. Soc.* **22B**, 353–367 (1981).
18. G. M. L. Gladwell, J. R. Barber and Z. Olesiak, Thermal problems with radiation boundary conditions, *Q. J. Mech. Appl. Math.* **36**(3), 387–401 (1983).
19. M. M. Yovanovich, Thermal constriction resistance of contacts on a half-space: integral formulation. In *AIAA Progress in Astronautics and Aeronautics, Radiative Transfer and Thermal Control*, Vol. 49, pp. 397–418. MIT, Cambridge, Massachusetts (1976).
20. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*. Oxford University Press, London (1959).
21. N. I. Ioakimidis and P. S. Theocaris, On the numerical solution of singular integrodifferential equations, *Q. Appl. Math.* **37**, 325–331 (October 1979).
22. F. Erdogan and G. D. Gupta, On the numerical solution of singular integral equations, *Q. Appl. Math.* **30**, 525–534 (1972).
23. J. R. Barber, Some thermoelastic contact problems involving frictional heating, *Q. J. Mech. Appl. Math.* **XXIX**(1), 1–13 (1976).
24. L. V. Kantorovich and V. I. Krylov, *Approximate Methods of Higher Analysis* (English translation by C. D. Benster). Interscience, New York (1958).
25. T. F. Lemczyk, Constriction resistance of axisymmetric convective contacts on semi-infinite domains, M.A.Sc. Thesis, University of Waterloo, Waterloo, Canada (1986).
26. M. M. Yovanovich, Thermal contact correlations, *AIAA Prog. Astronaut. Aeronaut.* **83**, 83–95 (1982).
27. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products.* Academic Press, New York (1980).
28. G. M. L. Gladwell and T. F. Lemczyk, Thermal constriction resistance of a contact on a circular cylinder with mixed convective boundaries, *Proc. R. Soc. London* (1988), in press.

APPENDIX: SOLUTION DETAILS

In the following, $[I]$ denotes the identity matrix, and $[N]$ is a diagonal matrix with entries $(2i-1)^{-1}$ for $i = 0, 1, \ldots, n$. Also, truncation size $n_T = n+1$.

Imperfect contacts (H_1 uniform)

$$\{[I]+H_1[N][D]\}a_i = g_i. \tag{A1}$$

$[D]$ is symmetric with entries given by equations (3.17) and (4.11) in ref. [18], and for both cases of external insulation or $T = 0$, we have

$$g_0 = \frac{2}{\pi}H_1\Theta_b, \quad g_n = 0, \quad n = 1, 2, \ldots \tag{A2}$$

Isothermal or flux-specified contacts (uniform H_2)

$$\{[C]+H_2[H][B]^{-1}[F]\}\alpha_i = -g_i. \tag{A3}$$

Matrix $[C]$ is *lower bi-diagonal*, with entries of 1/2. $[F]$ is symmetric with entries given by

$$f_{m,n} = \frac{1}{\pi}\left(\frac{1}{2m+2n-1} - \frac{1}{2m+2n+1} + \frac{1}{2m-2n+1} - \frac{1}{2m-2n-1}\right). \tag{A4}$$

$[B]$ is *upper bi-diagonal*, with entries given by

$$B_{i,i} = \frac{i+1}{2i+1}, \quad B_{i,i+1} = \frac{i+1}{2i+3}, \quad B_{n,n} = \frac{n+1}{2n+1}, \quad i = 0, 1, \ldots \tag{A5}$$

It's inverse is *upper triangular*, with explicit entries

$$B_{i,i}^{-1} = \frac{1}{B_{i,i}}, \quad B_{i,j}^{-1} = \frac{-B_{j-1,j}B_{i,j-1}^{-1}}{B_{j,j}}, \quad j = i+1, i+2, \ldots, n. \tag{A6}$$

For an isoflux or isothermal contact, g_i/q_0 are defined in ref. [18], and for a non-uniform flux contact of the form of equation (19), with arbitrary constants c and d, from ref. [25] we have

$$g_i = dg_i^1 + \frac{4cq_0}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m h_{m+n}}{m+n+1} \tag{A7}$$

where the g_i^1 are as defined for the isoflux case. The h_{m+n} are now given by

$$h_{m+n} = v_{m+n} - v_{m+n-1} + \cdots + (-1)^{m+n}v_0 \tag{A8}$$

and

$$v_m = r_m - \frac{8}{3\pi}. \tag{A9}$$

The r_m are found by matching coefficients in

$$\sum_{n=0}^{\infty}\frac{c_n}{2^n}(w)^n = \frac{\pi}{2}\sum_{n=0}^{\infty}r_m P_m(w) \tag{A10}$$

where P_m are the Legendre polynomials, and the c_n are the power series expansion coefficients for

$$\frac{\sin^{-1} w}{w^3} - \frac{(1-w^2)^{1/2}}{w^2}. \tag{A11}$$

Non-uniform contact conductance

With the convection coefficient $h(\rho)$ represented by equation (36), we have

$$\left\{[I]+H_1\left(d+\frac{c}{2}\right)[N][D]+H_1\frac{c}{2}[N][G][D']\right\}a_n = g_n. \tag{A12}$$

$[I]$, $[D]$ and $[N]$ are as defined for the uniform conductance problems, with $[D']$ having entries of $[D]$ truncated to size $(n_T+1)\times n_T$. The rectangular $(n_T\times(n_T+1))$ matrix $[G]$ has entries from

$$\begin{Bmatrix} b'_0 \\ b'_1 \\ \vdots \\ \vdots \\ b'_n \end{Bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & 0 & \cdots & & \\ 1 & 0 & \frac{2}{5} & \cdots & & \\ 0 & \frac{2}{3} & 0 & \frac{3}{7} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{n}{2n-1} & 0 & \frac{n+1}{2n+3} \end{bmatrix} \times \begin{Bmatrix} b_0 \\ b_1 \\ \vdots \\ \vdots \\ \vdots \\ b_{n+1} \end{Bmatrix} \quad \text{(A13)}$$

or simply

$$b'_n = [G]b_n. \quad \text{(A14)}$$

The g_n are

$$g_0 = \frac{2}{\pi}H_1\Theta_b\left(d+\frac{c}{2}\right), \quad g_1 = -\frac{H_1}{\pi}\Theta_b\left(\frac{c}{3}\right)$$

and for

$$n = 2,3,\ldots,n_T, \quad g_n = 0. \quad \text{(A15)}$$

Non-uniform external convection

With constants c and d defined now in equation (47), for either a flux-specified or an isothermal contact, the resulting matrix equation becomes

$$\left\{[C]+H_2[N]\left(\left(d+\frac{c}{2}\right)[B]^{-1}[F]+\frac{c}{2}[G][F'][C]\right)\right\}\alpha = g. \quad \text{(A16)}$$

For flux-specified contacts, the g_n are as defined for an *isoflux* contact, and in the form of equation (A7) for a *varying flux* contact. For an *isothermal* contact, the right-hand vector is determined from

$$\frac{H_2}{s^2}d\sum_{m=0}^{\infty} l_m P_m(2s^2-1) - 1 + H_2 c \sum_{m=0}^{\infty} l_m P_m(2s^2-1)$$
$$= \frac{\pi}{2}\sum_{n=0}^{\infty}(2n+1)(g_n^{(1)}+g_n^{(2)})P_n(2s^2-1) \quad \text{(A17)}$$

where

$$g_n^{(1)} = \frac{4}{\pi}H_2 d\sum_{m=0}^{\infty}\frac{(-1)^m f_{m+n}}{m+n+1} - \frac{2}{\pi}\delta_{n,0}$$
$$g_n^{(2)} = \frac{2}{\pi}H_2 c l_n (2n+1)^{-1}. \quad \text{(A18)}$$

The series term above is already defined for isothermal contact with the uniform external convection coefficient, here including the multiplying factor d. The l_n are defined by

$$l_n = c_n - d_n \quad \text{(A19)}$$

where

$$c_n = \frac{4}{\pi}(-1)^n\left[\frac{\pi}{2} - 2\left(1-\frac{1}{3}+\frac{1}{5}-\cdots+\frac{(-1)^n}{2n+1}\right)+\frac{(-1)^n}{2n+1}\right] \quad \text{(A20)}$$

$$d_n = -\frac{4}{\pi(2n+3)(2n-1)}. \quad \text{(A21)}$$

RESISTANCE THERMIQUE DE CONSTRICTION AVEC DES CONDITIONS AUX LIMITES CONVECTIVES—1. CONTACT DE DEMI-ESPACE

Résumé—L'analyse analytique du contact thermique a été dans le passé plutôt réduite à des conditions aux limites de contact. Récemment, Gladwell *et al.* (*Q. J. Mech. Appl. Math.* **36**(3), 387–401 (1983)) a dégagé l'évaluation effective de l'équation integro-différentielle pour quatre problèmes axisymétriques avec des conditions aux limites convectives uniformes. On étudie ici, sous forme adimensionnelle, la variation de la résistance thermique de constriction avec le nombre de Biot pour quatre types de problèmes mixtes sur un demi-espace homogène. De plus l'analyse thermique est élargie pour inclure des conditions de flux non uniforme et de convection non-uniforme. Dans chaque cas, la résistance de constriction est donnée sous forme d'une expression compacte et pour quelques cas des formules précises mais simples sont présentées.

DER THERMISCHE WIDERSTAND BEI KONVEKTIVEN RANDBEDINGUNGEN—1. KONTAKTE AN EINEM HALBRAUM

Zusammenfassung—Die Untersuchung des thermischen Kontakts ist in der Vergangenheit auf stark idealisierte Randbedingungen an der Kontaktfläche beschränkt gewesen. Vor kurzem haben Glaswell *et al.* (*Q. J. Mech. Appl. Math.* **36**(3), 387–401 (1983)) eine effiziente Auswertung der resultierenden Integral-Differential-Gleichung für vier grundlegende achsensymmetrische Probleme mit einheitlichen konvektiven Randbedingungen vorgestellt. Die vorliegende Arbeit zeigt in dimensionsloser Form die Veränderung des thermischen Widerstandes mit der Biot-Zahl für vier unterschiedliche Problemtypen an einem homogenen Halbraum auf. Zusätzlich wurde die thermische Analyse erweitert, um ungleichförmige Strömungen und ungleichförmige konvektive Bedingungen einzubeziehen. In allen Fällen erhält man den Widerstand als kompakten Ausdruck, und für mehrere Fälle werden genaue, aber einfachere Korrelationen vorgelegt.

ТЕРМИЧЕСКОЕ СОПРОТИВЛЕНИЕ ПРИ СЖАТИИ ДЛЯ КОНВЕКТИВНЫХ ГРАНИЧНЫХ УСЛОВИЙ—1. КОНТАКТЫ В ПОЛУПРОСТРАНСТВЕ

Аннотация—Ранее теоретический анализ тепловых контактов проводился при идеализированных условиях на контактной поверхности. Недавно Глэдвеллом и др. (*Q. J. Mech. Appl. Math.* **36**(3), 387–401 (1983)) был предложен эффективный метод решения интегро-дифференциального уравнения для четырех основных осесимметричных задач с однородными конвективными граничными условиями. В настоящей работе в безразмерном виде описывается зависимость термического сопротивления при сжатии от числа Био для четырех задач смешанного типа на однородном полупространстве. Кроме того, анализ распространен на неоднородный тепловой поток и на неоднородные конвективные условия. В каждом из них сопротивление сжатию описывается компактным выражением, а для нескольких случаев даны точные и более простые обобщающие соотношения.